**STAT0041: Stochastic Calculus**

# Lecture 3 - Discrete-time Martingale

Lecturer: Weichen Zhao — Fall 2024

**Key concepts**:

- *Martingale;*
- *Stopping time.*

## 3.1 Definition of martingale

**Example 3.1 (Fair Gambling)** *In a simple gambling, one decides the amount of the next game based on the results of the previous games. Let $\xi_0$ be the capital at the beginning of the game, and denote $\xi_n$ as all the principal after the nth game.*

*Let*

$$\eta_n = \begin{cases} 1 & \text{win } n\text{-th game;} \\ -1 & \text{lose } n\text{-th game.} \end{cases}$$

*be i.i.d. sequence of random variables satisfies $\mathrm{P}(\eta_n = 1) = p$, $\mathrm{P}(\eta_n = -1) = 1 - p = q$, and borel function $f_n(\xi_0, \eta_1, \eta_2, \cdots, \eta_{n-1})$ be strategy for the n-th game. Then, the principal after the n-th game is*

$$\xi_n = \xi_{n-1} + f_n(\xi_0, \eta_1, \eta_2, \cdots, \eta_{n-1}) \cdot \eta_n = \xi_0 + \sum_{k=1}^{n} f_k(\xi_0, \eta_1, \eta_2, \cdots, \eta_{k-1}) \cdot \eta_k.$$

*Consider*

$$\mathbb{E}[\xi_{n+1} | \xi_0, \eta_1, \cdots, \eta_n] = \xi_n + f_{n+1}(\xi_0, \eta_1, \cdots, \eta_n)\mathbb{E}(\eta_{n+1}).$$

*is the expectation/prediction of principal after the $(n+1)$-th game when one knows results of the previous n games. We want the game to be fair, that means, at any time n, predicting the "win/lose situation" at time $n+1$, regardless of the game strategy, is impossible to obtain any information about the "win/lose situation". That is*

$$\mathbb{E}[\xi_{n+1} | \xi_0, \eta_1, \cdots, \eta_n] = \xi_n.$$

*When $p = q = \frac{1}{2}$, we realize "fair gambling".*

This leads to the following definition

**Definition 3.2 (Martingale)** *Let $(\Omega, \mathscr{F}, (\mathscr{F}_n), \mathrm{P})$, $n = 0, 1, \ldots$ be a filtered probability space, $X = (X_n)$ be a adapted process on $(\Omega, \mathscr{F}, (\mathscr{F}_n), \mathrm{P})$ satisfying $\mathbb{E}[|X_n|] < \infty$ is called*

*(1) a $\mathscr{F}_n$-**martingale** if $\mathbb{E}[X_{n+1} | \mathscr{F}_n] = X_n$;*

*(2) a $\mathscr{F}_n$-supermartingale if $\mathbb{E}[X_{n+1} | \mathscr{F}_n] \le X_n$;*

*(3) a $\mathscr{F}_n$-submartingale if $\mathbb{E}[X_{n+1} | \mathscr{F}_n] \ge X_n$.*

## 3.2 Examples

**Example 3.3 (Doob martingale)** *Let* $(\Omega, \mathscr{F}, (\mathscr{F}_n), \mathrm{P})$, $n = 0, 1, \ldots$ *be a filtered probability space,* $\xi$ *be a random variable satisfies* $\mathbb{E}|\xi| < \infty$. *Define* $\xi_n = \mathbb{E}[\xi|\mathcal{F}_n]$, *then* $\xi_n$ *is a martingale.*

**Example 3.4 (Martingale transform)** *For* $(\Omega, \mathscr{F}, (\mathscr{F}_n), \mathrm{P})$, $n = 0, 1, \ldots$, *let* $(C_n)$, $n = 0, 1, \ldots$ *be a sequence of random variables. We say* $(C_n)$ *is* ***predictable****, if* $C_n$ *is* $\mathscr{F}_{n-1}$*-measurable for all* $n \geq 1$.

*Let* $(X_n)$ *be a* $\mathscr{F}_n$*-martingale,* $(C_n)$ *be a* $\mathscr{F}_n$*-predictable process. Define* martingale transform *of* $(X_n)$ *respect to* $(C_n)$*:*

$$Y_n := \sum_{k=1}^{n} C_k(X_k - X_{k-1}),\ k \geq 1, \quad Y_0 = 0. \tag{3.1}$$

*Then* $Y_n$ *is a* $\mathscr{F}_n$*-martingale.*

*In practice, we consider a risk investment in the market with price* $(X_n)$ *and a fixed income with interest rate* $r$. *Someone have initial fortune* $Y_0$. *An investment strategy is deciding at* $n-1$ *to hold* $C_n$ *risk investment at* $n$, *and with the remaining funds purchasing fixed income. The fortune at* $n-1$ *is*

$$Y_{n-1} = C_n X_{n-1} + (Y_{n-1} - C_n X_{n-1}).$$

*Then at* $n$

$$\begin{aligned} Y_n &= C_n X_n + (1+r)(Y_{n-1} - C_n X_{n-1}) \\ \iff Y_n - (1+r)Y_{n-1} &= C_n(X_n - (1+r)X_{n-1}) \\ \iff (1+r)^{-n}Y_n - (1+r)^{-(n-1)}Y_{n-1} &= C_n[(1+r)^{-n}X_n - (1+r)^{-(n-1)}X_{n-1}]. \end{aligned}$$

*Discount fortune* $(1+r)^{-n}Y_n$ *is martingale transform of discount price* $(1+r)^{-n}X_n$ *respect to* $C_n$.

**Example 3.5 (Martingale betting strategy)** *Considering gambling model in example 3.1, we change the strategy to that each bet is doubled accordingly from the previous bet before the first win. This strategy is called martingale betting strategy. Specifically,*

$$\begin{gathered} f_1(\xi_0) = 1, \\ f_2(\xi_0, -1) = 2,\ f_2(\xi_0, 1) = 0, \\ f_3(\xi_0, -1, -1) = 4,\ f_3(\xi_0, -1, 1) = 0,\ f_3(\xi_0, 1, -1) = 0,\ f_3(\xi_0, 1, 1) = 0, \ldots \\ f_n(\xi_0, \underbrace{-1, \cdots, -1}_{n-1}) = 2^{(n-1)},\ f_n(\xi_0, others) = 0. \end{gathered}$$

*Since* $p = q = \frac{1}{2}$, $(\xi_n)$ *is a martingale. Let*

$$A_n := \{\eta_1 = -1, \cdots, \eta_{n-1} = -1, \eta_n = 1\}$$

*denote the event that the gambler wins for the first time at* $n$. *Then*

$$A_n = \{\xi_0 = 0, \xi_1 = -1, \cdots, \xi_{n-1} = -\sum_{k=1}^{n-1} 2^{k-1}, \xi_n = \xi_{n-1} + 2^{n-1}, \xi_{n+k} = \xi_n, k \geq 1\}.$$

*Event* $\bigcup_{n=1}^{\infty} A_n$ *denote gambler wins and stops gambling sooner or later.*

$$\mathrm{P}(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathrm{P}(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

*This shows that even in the case of a fair gambling, there is "martingale betting strategy" that guarantees the gambler will not lose in the end. Does this contradict the fact that gambling with martingale property is "fair"?*

## 3.3 Stopping time

In life we care about special moments like births, graduations, and marriages. But for different people, the moment of the same event may be different. This motivates us to consider not only deterministic time such as 0, 1, 2, etc., but also random time. In stochastic processes, we introduce the concept of stopping time which is a special class of random time.

**Definition 3.6 (Stopping time)** *Let $\tau$ be a random variable, which is defined on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n), \mathrm{P})$ with values in $\mathbb{N} \cup \{+\infty\}$ . Then $\tau$ is called a* ***stopping time*** *(with respect to the filtration $(\mathscr{F}_n)$), if the following condition holds:*

$$\{\omega \in \Omega : \tau(\omega) \leq n\} \in \mathscr{F}_n \text{ for all } n.$$

*Constant $n$ is constant stopping time.* ***Stopped process*** *$X^\tau$ is defined as*

$$X_n^\tau(\omega) := X_{\tau(\omega) \wedge n}(\omega)$$

Intuitively, this condition means that the "decision" of whether to stop at time $n$ must be based only on the information present at time $n$, not on any future information.

**Proposition 3.7** *Let $\tau$ and $\sigma$ be two $(\mathscr{F}_n)$ stopping times, then*

*(1) $\tau \wedge \sigma := \min(\tau, \sigma)$ is a $(\mathscr{F}_n)$ stopping time;*

*(2) $\tau \vee \sigma := \max(\tau, \sigma)$ is a $(\mathscr{F}_n)$ stopping time.*

For filtration $(\mathscr{F}_n)$, event field $\mathscr{F}_n$ is the information before time $n$. For a stopping time $\tau$, we want to know what is the "information" that can be held before $\tau$. We introduce the following definition:

**Definition 3.8 (Event field of $\tau$-past)** *Let $\tau$ be a stopping time on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n), \mathrm{P})$. Then*

$$\mathscr{F}_\tau := \{A \in \mathscr{F} \mid \forall n,\ \{\omega \in \Omega : \tau(\omega) \leq n\} \cap A \in \mathscr{F}_n\} \tag{3.2}$$

*is called the* ***event field of $\tau$-past****.*

**Proposition 3.9** *Let $\tau$ and $\sigma$ be two $(\mathscr{F}_n)$ stopping times, then*

*(1) $\tau$ is $\mathscr{F}_\tau$ measurable;*

*(2) If $\tau \leq \sigma$, then $\mathscr{F}_\tau \subset \mathscr{F}_\sigma$.*